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MAGNETIC FIELD EFFECTS IN TRAVELING WAVE TUBES

by

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ABSTRACT

The operation of traveling-wave tubes is affected by the DC magnetic field used for focusing the beam. A full theoretical analysis of this effect is herein presented. The Maxwell equations are solved for the field configuration of the modes of the system. The cylindrical sheath boundary conditions are applied at the helix, and appropriate conditions at the beam radius. A transcendental equation is derived whose roots are the propagation constants of the modes. This equation is reduced by an approximation procedure to a quintic. It is proven that this quintic has at least three real roots, so no new amplified waves are generated. A numerical analysis of the roots of the quintic will appear in a future report.

I. Introduction

In the normal manner of operation, the electron beam of a traveling wave tube is constrained by a static longitudinal magnetic field. Most previous analyses of the tube have assumed that this magnetic field is infinitely strong, so that all radial motion of the electrons is suppressed. In actuality some radial motion does occur, and its effect is most marked at very high frequencies. In this paper a full theoretical investigation of the magnetic field effects will be carried out. Numerical analysis of the results will be deferred to a later paper. A distortion of the gain frequency curve has actually been noted experimentally[1], [2].

A partial analysis of the magnetic field effects has been developed by Pierce[3]. He assumes that the electron beam is ribbon-shaped, so that the motion takes place in one longitudinal and one transverse coordinate. With this and several other assumptions, he derives for the modes of the system an equation which is of the fifth degree instead of the usual cubic. However, the coefficients in this quintic equation involve an unknown parameter, which is the ratio of the transverse electric field to the longitudinal. This parameter can only be fixed by comparison with experiment.

This paper starts with the ten equations which govern the behavior of the system in cylindrical coordinates. It is assumed that there is no angular dependence, and the standard cylindrical sheath assumption is used to simplify the boundary conditions. In Section II, the ten equations are solved to obtain the functional form of the system variables. In Section III, the boundary conditions are applied to obtain the transcendental equation satisfied by the propagation constants of the modes of the system. Up to this point the effect of the AC magnetic field of the beam has been included. A "non-relativistic transition" is now performed, in which terms of the order of magnitude of $(v/c)^2$, where v is the electron velocity, are dropped. This results in a much simplified transcendental equation.

In Section IV, this transcendental equation is reduced to an algebraic equation, using the method developed by Friedman[4]. The resulting equation is also a quintic, which may be rendered identical with Pierce's by a suitable nomenclature. However, the arbitrary feature of Pierce's theory is removed, since all the coefficients of the quintic are derived here from fundamental field theory. The necessary numerical analysis and comparison with experiment will appear in a future report, but it is proven that no new amplified waves are generated.

II. Solution of the Field Equations

The system is governed by ten equations. First there are the six Maxwell equations. Then there are the three Lorentz force equations, and finally the equation of continuity. The three equations relating current, charge, and electron velocity may be viewed as definitions of the current components, and only add redundancy to the system. In these equations the variables, which will be denoted by circumflexed letters, represent total field variables, including both AC and DC components. The equations, in MKS units, are:

$$\nabla \times \underrightarrow{\hat{E}} + \mu \frac{\partial}{\partial t} \underrightarrow{\hat{H}} = 0 \tag{2.1a}$$

$$\nabla \times \underrightarrow{\hat{H}} - \epsilon \frac{\partial}{\partial t} \underrightarrow{\hat{E}} = \hat{\rho}\, \underrightarrow{\hat{v}} \tag{2.1b}$$

$$m \frac{d}{dt} \underrightarrow{\hat{v}} = m\left(\frac{\partial}{\partial t} + \underrightarrow{\hat{v}} \cdot \nabla\right) \underrightarrow{\hat{v}} = e\left(\underrightarrow{\hat{E}} + \mu\, \underrightarrow{\hat{v}} \times \hat{H}\right) \tag{2.1c}$$

$$\nabla \cdot \hat{\rho} \hat{v} + \frac{\partial}{\partial t} \hat{\rho} = 0 \,. \tag{2.1d}$$

These equations may be separated into DC and AC components. It will be assumed that in the absence of AC fields, the electron beam is perfectly collimated and of cylindrical cross-section of radius b. Furthermore, the presence of a small amount of residual gas in the tube will effectively neutralize the fields produced by the DC space charge in the beam. The entire tube will be immersed in a static magnetic field of strenghth H_o directed along the axis of the tube. Since there is no initial radial or angular motion, none will develop if DC space charge forces are neglected. All DC quantities will be denoted by a subscripted zero, while AC quantities will be denoted by primed letters and a factor $e^{j\omega t}$. Since the equations contain nonlinear terms, harmonics will be produced in the beam. These will be neglected; hence the theory presented in this report will be restricted to a small signal analysis. The separation is accordingly:

$$\underrightarrow{\hat{E}} = \underrightarrow{E'}\, e^{j\omega t} \tag{2.2a}$$

$$\underrightarrow{\hat{H}} = H_o\, \underrightarrow{i_z} + \underrightarrow{H'}\, e^{j\omega t} \tag{2.2b}$$

$$\rho = \rho_o + \rho' e^{j\omega t} \tag{2.2c}$$

$$\underrightarrow{\hat{v}} = v_o\, \underrightarrow{i_z} + \underrightarrow{v'}\, e^{j\omega t} . \tag{2.2d}$$

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$$\underrightarrow{\hat{v}} = v_o \underrightarrow{i_z} + \underrightarrow{v}' e^{j\omega t}. \tag{2.2d}$$

is spoiled here by the magnetic coupling terms $h_o v_r$ and $h_o v_\theta$ in the force equation, which terms cause the two types of waves to be coupled. The result of this elimination of transverse fields is:

$$E_r = \frac{j}{p^2}\left[q\,\frac{\partial E_z}{\partial r} - k\beta h^2 v_r\right] \tag{2.8a}$$

$$E_\theta = -\frac{jk}{p^2}\left[\frac{\partial H_z}{\partial r} + \beta h^2 v_\theta\right] \tag{2.8b}$$

$$H_r = \frac{jq}{p^2}\left[\frac{\partial H_z}{\partial r} + \beta h^2 v_\theta\right] \tag{2.8c}$$

$$H_\theta = \frac{j}{p^2}\left[k\,\frac{\partial E_z}{\partial r} - q\beta h^2 v_r\right] \tag{2.8d}$$

$$p^2 = q^2 - k^2\,. \tag{2.8e}$$

Next, these results are substituted into Equations (2.7h) and (2.7i). This yields two equations which enable us to express v_r and v_θ in terms of the longitudinal field components E_z and H_z, or rather their radial derivatives. Also, Equation (2.7j) enables us to express v_z in terms of E_z. The determinant of the v_r, v_θ equations will occur sufficiently often to warrant a special symbol; D will be used. This determinant is given by:

$$D = h_o^2 - (k'-q)^2\left(1 + \frac{\beta^2 h^2}{p^2}\right)^2. \tag{2.9}$$

In terms of D the solution for the velocities is given by:

$$v_r = \frac{j(k'-q)}{Dp^2}\left[j(q-k'\beta^2)\left(1+\frac{\beta^2 h^2}{p^2}\right)\frac{\partial E_z}{\partial r} - \beta h_o\,\frac{\partial H_z}{\partial r}\right] \tag{2.10a}$$

$$v_\theta = -\frac{j}{Dp^2}\left[j\beta(k'-q)\left(1+\frac{\beta^2 h^2}{p^2}\right)\frac{\partial H_z}{\partial r} + h_o\,(q-k'\beta^2)\,\frac{\partial E_z}{\partial r}\right] \tag{2.10b}$$

$$v_z = -\frac{j}{k'-q}\,E_z\,. \tag{2.10c}$$

With the velocities now obtained, next the charge density may be found from Equation (2.7g). The result of this simple calculation is

$$\rho = -\frac{jq}{(k'-q)^2}\,E_z - \frac{1}{Dp^2}\left[j(q-h'\beta^2)\left(1+\frac{\beta^2 h^2}{p^2}\right)\frac{1}{r}\frac{\partial}{\partial r}\,r\frac{\partial}{\partial r}\,E_z - \beta h_o\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}\,H_z\right]. \tag{2.11}$$

Up to this point the transverse fields and all the electronic components have been expressed in terms of the longitudinal fields. We are now ready for the grand step of substituting these results into the remaining field equations (2.7c) and (2.7f). This will yield wave equations for the system. The coefficients in these wave equations are sufficiently complicated that they also deserve special symbols. Therefore, define:

$$C_1 = h_o^2 - (k'-q)^2\left(1 + \frac{\beta^2 h^2}{p^2}\right) \tag{2.12a}$$

$$C_2 = \beta h^2 h_o (q-k'\beta^2)/p^2 \tag{2.12b}$$

$$C_3 = h_o^2 - \left\{(k'-q)^2 - h^2(1-\beta^2)\right\}\left(1 + \frac{\beta^2 h^2}{p^2}\right) \tag{2.12c}$$

$$\eta^2 = p^2\left[1 - \frac{h^2}{(k'-q)^2}\right]. \tag{2.12d}$$

The last symbol, η, has been used by other writers to represent the "radial propagation constant" within the beam. Using these symbols and skipping the algebraic steps necessary in the reduction, we obtain the following wave equations for the system:

$$C_1 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} H_z - Dp^2 H_z = jC_2 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} E_z \tag{2.13a}$$

$$C_3 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} E_z - D\eta^2 E_z = -jC_2 \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} H_z . \tag{2.13b}$$

Thus, we have two coupled second order differential equations for the longitudinal field components. Although partial derivative signs appear, the functions E_z and H_z which appear from Equation (2.7) on depend only on r, since both the θ and z dependence have been factored out. The equations are thus actually ordinary differential equations.

The Equations (2.13a) and (2.13b) may be solved very simply. Try the solution:

$$E_z = AI_o(\alpha r) \tag{2.14a}$$

$$H_z = BI_o(\alpha r). \tag{2.14b}$$

The possible other solution $K_o(\alpha r)$ is rejected, since we are dealing with a solid cylindrical beam and $K_o(\alpha r)$ becomes infinite at r = 0. This other solution must of course be kept outside the beam. When the trial solution (2.14) is inserted into the differential equations (2.13), the following pair of algebraic equations results:

$$(C_1\alpha^2 - Dp^2)\, B - jC_2\alpha^2\, A = 0 \tag{2.15a}$$

$$jC_2\alpha^2 B + (C_3\alpha^2 - D\eta^2)\, A = 0. \tag{2.15b}$$

In order that these equations have a solution other than A=B=0, their determinant must vanish. Accordingly there results for α the fourth degree equation:

$$(C_1C_3 - C_2^2)\alpha^4 - D(C_1\eta^2 + C_3p^2)\alpha^2 + D^2p^2\eta^2 = 0. \tag{2.16}$$

This may be solved directly, yielding two possibilities for α^2:

$$\alpha_1^2 = D\left[C_1\eta^2 + C_3p^2 + \left\{(C_1\eta^2 - C_3p^2)^2 + 4C_2^2p^2\eta^2\right\}^{1/2}\right]/2(C_1C_3 - C_2^2) \tag{2.17a}$$

$$\alpha_2^2 = D\left[C_1\eta^2 + C_3p^2 - \left\{(C_1\eta^2 - C_3p^2)^2 + 4C_2^2p^2\eta^2\right\}^{1/2}\right]/2(C_1C_3 - C_2^2). \tag{2.17b}$$

These expressions are purely real. Since the Bessel function $I_o(\alpha r)$ involves only even powers of α, only the positive signs need be taken in finding α from α^2, since the negative signs do not yield new solutions.

Accordingly, for a given value of q, there are two corresponding values for the radial propagation constant. The situation is somewhat analogous to the situation in the customary theory, in which E_z has the radial propagation constant η, while H_z has the radial propagation constant p. It is easily shown that in the limit $h_o \to \infty$, which corresponds to the customary theory, α_1 approaches η, and α_2 approaches p. For finite values of h_o, a mixture of the two solutions must be used, which of course is appropriate to the coupled system. Accordingly, there are two arbitrary constants in the solution, which may be used to satisfy the boundary conditions.

It is now possible to write out all the field components in terms of the radial propagation constants α_1 and α_2 and the two arbitrary constants, which will be taken to be the coefficients of $I_o(\alpha_1 r)$ and $I_o(\alpha_2 r)$ in E_z. This provides a solution to the field equations within the beam. Outside the beam, a solution of the charge-free field equations must be employed. To save very much writing, only that part of the interior solution which involves α_1 will be written down. Hence, to the solution appearing in the following Equations (2.18) must be added a solution obtained by replacing α_1 everywhere by α_2, A_1 by A_2, and keeping everything else unchanged. A subscripted i denotes inside the beam. The solution is:

$$E_{zi} = A_1 I_o(\alpha_1 r) \tag{2.18a}$$

$$E_{ri} = \frac{j}{p^2}\left[q + \frac{k'\beta^2h^2(k'-q)(q-k'\beta^2)}{p^2}\,\frac{\alpha_1^2-p^2-\beta^2h^2}{C_1\alpha_1^2-Dp^2}\right]\alpha_1 A_1 I_1(\alpha_1 r) \tag{2.18b}$$

$$E_{\theta i} = \frac{k'\beta C2}{C_1\alpha_1^2-Dp^2}\;\alpha_1 A_1 I_1(\alpha_1 r) \tag{2.18c}$$

$$H_{zi} = \frac{jC_2\alpha_1^2}{C_1\alpha_1^2-Dp^2} \quad A_1 I_o(\alpha_1 r) \tag{2.18d}$$

$$H_{ri} = -\frac{qC_2}{C_1\alpha_1^2-Dp^2} \; \alpha_1 A_1 I_1(\alpha_1 r) \tag{2.18e}$$

$$H_{\theta i} = \frac{j\beta}{p^2}\left[k' + \frac{qh^2(k'-q)(q-k'\beta^2)}{p^2} \; \frac{\alpha_1^2-p^2-\beta^2h^2}{C_1\alpha_1^2-Dp^2}\right] \alpha_1 A_1 I_1(\alpha_1 r) \tag{2.18f}$$

$$v_{zi} = -\frac{j}{k'-q} \; A_1 I_o(\alpha_1 r) \tag{2.18g}$$

$$v_{ri} = -\frac{(k'-q)(q-k'\beta^2)}{p^2} \; \frac{\alpha_1^2-p^2-\beta^2h^2}{C_1\alpha_1^2-Dp^2} \quad \alpha_1 A_1 I_1(\alpha_1 r) \tag{2.18h}$$

$$v_{\theta i} = -\frac{jh_o(q-k'\beta^2)(\alpha_1^2-p^2)}{p^2(C_1\alpha_1^2-Dp^2)} \quad \alpha_1 A_1 I_1(\alpha_1 r) \tag{2.18i}$$

$$\rho_i = -j\left[\frac{q}{(k'-q)2} + \frac{(q-k'\beta^2)}{p^2} \; \frac{\alpha_1^2-p^2-\beta^2h^2}{C_1\alpha_1^2-Dp^2} \quad \alpha_1^2\right] A_1 I_o(\alpha_1 r) . \tag{2.18j}$$

With this solution and the associated one involving α_2 and A_2, the field quantities within the beam are determined. The solution (2.18) must be properly matched to a solution outside the beam. This latter solution may be found very quickly. Since boundary conditions are to be impressed at the beam radius, at the helix, and at infinity, two external solutions are needed: one to take care of the region between the beam radius and the helix, the other between the helix and infinity. These will be denoted by a subscripted m and o respectively. In region m, both kinds of Bessel functions must be used, but to satisfy the boundary conditions of finiteness at infinity only the K-type Bessel functions may be employed outside the helix, and the real part of p must be positive. The solutions are :

$$E_{zm} = A_3 I_o(pr) + A_4 K_o(pr) \qquad E_{zo} = A_5 K_o(pr) \tag{2.19a}$$

$$E_{rm} = \frac{jq}{p}\left[A_3 I_1(pr) - A_4 K_1(pr)\right] \qquad E_{ro} = -\frac{jq}{p} A_5 K_1(pr) \tag{2.19b}$$

$$E_{\theta m} = -\frac{jk}{p}\left[B_3 I_1(pr) - B_4 K_1(pr)\right] \qquad E_{\theta o} = \frac{jk}{p} B_5 K_1(pr) \tag{2.19c}$$

$$H_{zm} = B_3 I_0(pr) + B_4 K_0(pr) \qquad H_{zo} = B_5 K_0(pr) \tag{2.19d}$$

$$H_{rm} = \frac{jq}{p}\left[B_3 I_1(pr) - B_4 K_1(pr)\right] \qquad H_{ro} = -\frac{jq}{p} B_5 K_1(pr) \tag{2.19e}$$

$$H_{\theta m} = \frac{jk}{p}\left[A_3 I_1(pr) - A_4 K_1(pr)\right] \qquad H_{\theta o} = -\frac{jk}{p} A_5 K_1(pr) \tag{2.19f}$$

Thus, the solutions (2.18) and (2.19) taken together contain eight arbitrary constants, which must be determined from boundary conditions. Because of the redundancy of the Maxwell equations, there will be eight linearly independent boundary conditions to impose. Since these conditions are all homogeneous, a determinantal equation must be satisfied by the propagation constant. The satisfying of the boundary conditions will form the body of the next section.

III. Application of the Boundary Conditions

It is necessary to impress four boundary conditions at the helix, and four conditions at the beam. Let the pitch of the helix be ψ. The helix conditions will be taken to be the standard "helical sheath" conditions [6]. Thus, the component of electric field in the helical direction shall vanish, the component of electric field in the helical sheath perpendicular to the helical direction shall be continuous, and the component of magnetic field in the helical direction shall be continuous. Thereby are provided the four conditions, all for $r=a$:

$$E_{\theta m} \cos\psi + E_{zm} \sin\psi = 0 \tag{3.1a}$$

$$E_{\theta o} \cos\psi + E_{zo} \sin\psi = 0 \tag{3.1b}$$

$$-E_{\theta m} \sin\psi + E_{zm} \cos\psi = -E_{\theta o} \sin\psi + E_{zo} \cos\psi \tag{3.1c}$$

$$H_{\theta m} \cos\psi + H_{zm} \sin\psi = H_{\theta o} \cos\psi + H_{zo} \sin\psi\ . \tag{3.1d}$$

The first three of these equations imply the continuity of E_θ and E_z at $r=a$. Thereby the coefficients A_5 and B_5 are determined, and the outer region fields are given in terms of the middle region fields:

$$A_5 = A_3 \frac{I_0(pa)}{K_0(pa)} + A_4 \tag{3.2a}$$

$$B_5 = -B_3 \frac{I_1(pa)}{K_1(pa)} + B_4\ . \tag{3.2b}$$

Substitute these results in the fourth of Equations (3.1). When this is done, all terms involving A_4 and B_4 cancel from the equation, leaving a relation between A_3 and B_3. This may be solved for B_3:

$$B_3 = - \frac{jk\cot\psi}{p} \frac{K_1(pa)}{K_o(pa)} A_3 \,. \tag{3.3}$$

Finally, substitute this expression into the first of Equations (3.1). This produces a relation among A_3, A_4, and B_4. Define a function G(pa) by the equation

$$G(pa) = \frac{I_o(pa)K_o(pa) - \frac{k^2\cot^2\psi}{p^2} I_1(pa)K_1(pa)}{K_o^2(pa)} \,. \tag{3.4}$$

Then the result for B_4 takes on the simple form:

$$B_4 = - \frac{p}{jk\cot\psi} \frac{K_o(pa)}{K_1(pa)} \left[A_4 - G(pa)A_3 \right]. \tag{3.5}$$

The properties of the function G(pa) have been discussed by Friedman [4].

Up to this point, four of the eight arbitrary constants have been eliminated, leaving only A_1, A_2, A_3, and A_4. To determine these, it is necessary to impose conditions at the boundary of the beam. This immediately raises the question: just where is the boundary of the beam? Because of the action of the high-frequency space charge radial forces, which are here not completely suppressed by the static magnetic field, the beam will alternately expand and contract. A point near the undisturbed radius r=b will find itself inside the beam at certain times and outside the beam at other times. The analysis of such a situation is patently impossible. Accordingly, an approximate description of the state of affairs must be sought. This description was found during the course of an investigation of klystron theory by Feenberg [7].

Feenberg's basic idea is to make consistent use of the fact that the system is being treated by a small signal analysis. Consider an electron which extends at z=0, t=0 at the radius r=b. At the time t, it will have reached a position r,z, given by:

$$r = b + v_o \int_o^t v_r(r',z')\, e^{j\omega t'} dt' \tag{3.6a}$$

$$z = v_o t + v_o \int_o^t v_z(r',z') e^{j\omega t'} dt'. \tag{3.6b}$$

In Equations (3.6), the normalized velocities have already been introduced. The quantities r', z' imply that the integral in (3.6) is to be taken along the path of the electron, and its actual position r',z' at the time t' is to be put into the integral.

Now remember that the analysis is small signal. Therefore, in the arguments r', z' of v_r and v_z, r' may be replaced by b, and z' by $v_0 t'$, since thereby only second order terms are neglected. The z' dependence is as $e^{-jqz'}$, whence the effective radial displacement of the electron is given by:

$$r = b + v_r(b)\int_0^z e^{j(k'-q)z'}dz' \tag{3.7a}$$

$$= b + v_r(b)\left[e^{j(k'-q)z}-1\right]/j(k'-q). \tag{3.7b}$$

Thus, the undisturbed beam is surrounded by a ring of charge, in which the thickness of this ring, being proportional to $v_r(b)$, is a first order quantity. Boundary conditions should be impressed both on the inside and outside of the ring. However, in view of the small-signal assumption, the ring may be replaced by a surface charge at $r=b$, in which the amount of surface charge is the amount of charge between $r=b$ and r given by (3.7). With this surface charge is associated a surface current, of which only the z-component will be a first order quantity. The boundary conditions to be impressed are that the tangential electric and normal magnetic fields be continuous while the normal electric and tangential magnetic fields suffer discontinuities respectively proportional to the surface charge density and to the surface current density. The proportionality constants are determined by a consideration of the units in which the fields are measured. The resulting boundary conditions become, at $r=b$,

$$E_{zi} = E_{zm} \tag{3.8a}$$

$$E_{\theta i} = E_{\theta m} \tag{3.8b}$$

$$H_{ri} = H_{rm} \tag{3.8c}$$

$$H_{zi} = H_{zm} \tag{3.8d}$$

$$E_{rm} - E_{ri} = h^2 v_{ri}/j(k'-q) \tag{3.8e}$$

$$H_{\theta m} - H_{\theta i} = \beta h^2 v_{ri}/j(k'-q). \tag{3.8f}$$

The continuity of H_z arises from the fact that the quantity actually proportional to the surface current density is $\underset{\sim}{i}_r \times \underset{\sim}{H}$, where $\underset{\sim}{i}_r$ is the unit radial vector, and since the surface current density has only a z-component, only the θ-component of $\underset{\sim}{H}$ will be discontinuous. Now Equations (3.8b) and (3.8c) have the same mathematical content in view of the field Equation (2.7a) which is satisfied both inside and outside the beam. Similarly, the discontinuity in H_θ is determined from the discontinuities in E_r and v_r by the field Equation (2.7d), with which (3.8e) and (3.8f) are consistent. Hence (3.8f) provides no further restriction once (3.8e) is imposed. Thus the six Equations (3.8) are really only four equations, which can determine the four constants A_1, A_2, A_3, and A_4. It will be noted that a constant term in the radial displacement given by (3.7) has been omitted in writing the Equations (3.8). This constant term arises from initial conditions, which we have not attempted to satisfy. If it were kept, the modes of the system would have to satisfy a complicated "consistency condition", the effect of which has been treated for klystrons by Feenberg [7], and which produces only small corrections. Also, since we are mainly interested in amplified waves, the constant term in the radial displacement will be small in magnitude compared to the exponential term. The Equations (3.8) may accordingly be considered to represent the state of affairs at a considerable distance down the tube from the point of initiation of the electrons and the waves.

Some algebraic simplifications may be performed when the field solutions (2.18) and (2.19) are substituted into the Equations (3.8). In particular, the combination of radial electric field and radial velocity appearing in (3.8e) simplify very much. It is also convenient to introduce a function F(pa), defined by:

$$F(pa) = \frac{k\cot\psi}{p}\,\frac{K_1(pa)}{K_o(pa)} \quad . \tag{3.9}$$

This function $F(pa) = jE_z(a)/H_z(a)$, and hence represents an effective longitudinal impedance for the helical wire. Upon including the terms in A_2 in the field solutions (2.18), substituting into (3.8), and making the possible simplifications, we obtain the following four simultaneous equations for the four constants A_1, A_2, A_3, A_4:

$$A_1 I_o(\alpha_1 b) + A_2 I_o(\alpha_2 b) = A_3 I_o(pb) + A_4 K_o(pb) \tag{3.10a}$$

$$\frac{C_2\alpha_1 A_1 I_1(\alpha_1 b)}{C_1\alpha_1^2 - Dp^2} + \frac{C_2\alpha_2 A_2 I_1(\alpha_2 b)}{C_1\alpha_2^2 - Dp^2} = -\frac{FA_3 I_1(pb)}{p} - \frac{(A_4 - GA_3)K_1(pb)}{pF} \tag{3.10b}$$

$$\frac{C_2\alpha_1^2 A_1 I_o(\alpha_1 b)}{C_1\alpha_1^2 - Dp^2} + \frac{C_2\alpha_2^2 A_2 I_o(\alpha_2 b)}{C_1\alpha_2^2 - Dp^2} = -FA_3 I_o(pb) + \frac{(A_4 - GA_3)K_o(pb)}{F} \tag{3.10c}$$

$$\frac{\eta^2}{p}\left[\frac{A_1 I_1(\alpha_1 b)}{\alpha_1} + \frac{A_2 I_1(\alpha_2 b)}{\alpha_2}\right] = A_3 I_1(pb) - A_4 K_1(pb). \tag{3.10d}$$

The equivalence between (3.8) and (3.10) is that (3.10a) corresponds to (3.8a), (3.10b) to (3.8b), (3.10c) to (3.8d), and (3.10d) to (3.8e).

The Equations (3.10) are four simultaneous linear homogeneous equations for the four unknowns A_1, A_2, A_3, and A_4. For a nontrivial solution to exist, the determinant of the four equations must vanish. This provides a transcendental equation for the propagation constant q which determines the possible free modes of the system.

Although Equation (3.15) is much simpler than the full equation derived from the vanishing of the determinant of Equations (3.10), it is still a rather involved equation to deal with as far as complex roots are concerned. Hence, it would be desirable to reduce (3.15) to an algebraic equation. The method of doing this has been indicated by Friedman[4], who reduced Equation (3.16) to a cubic. In the next section, the method will be applied to (3.15), which will thereby be reduced to a quintic.

IV. Reduction of the Transcendental Equation

The basic idea of Friedman's method is to regard the roots of the transcendental equation in the presence of the beam as small perturbations of the roots in the absence of the beam. Hence, the roots are expanded in powers of the current, and only the lowest order terms are kept. This method will now be applied to Equation (3.15).

When the current is zero, Equation (3.15) reduces to $G(pa)=0$. This equation has one real positive root, and no other roots in the half-plane where the real part of p is positive. We may recall that the assumption that the real part of p is positive was made in writing the solution (2.19) for the field components outside the helix, since only for the real part of p being positive will the K-type of Bessel function decay exponentially at infinity. The one real positive root of $G(pa)=0$ will be called p_o, and $p_o a$ will be called X_o.

In the presence of the beam, the roots will depart only slightly from p_o. To save much writing, we shall introduce the following notation:

$$pa = X = X_o + \delta \tag{4.1a}$$

$$k'a = X' = X_o + d_o \tag{4.1b}$$

$$(ha)^2 = W \tag{4.1c}$$

$$h_o a = v \tag{4.1d}$$

$$b/a = u\ . \tag{4.1e}$$

Also, the longitudinal propagation constant q, which appears explicitly in α_1, will be approximated by p. This is consistent with the small velocity approximation that has been employed to derive (3.15), since only second order quantities are neglected. As far as the order of magnitude of the quantities in (4.1) are concerned, X_o, v, and u are of zero order, while δ and W are of first order. The quantity d_o will generally be small, since the tube is usually operated with the electron velocity close to the velocity of the unperturbed waves, but to achieve generality d_o will be treated as of zero order, since the dispersive character of the tube is of interest and the major frequency dependence of the roots is through d_o.

Multiply Equation (3.15) by $I_0(pb)/I_1(pb)$. The right hand side then differs from unity by a quantity of order δ, since $G(X)$ is of order δ. Expanding, the right side to first order in δ yields

$$\frac{\eta^2 I_1(\alpha_1 b)}{p\alpha_1 I_1(pb)}\;\frac{I_0(pb)}{I_0(\alpha_1 b)} = 1 - \frac{\delta G'(X_0)}{uX_0 I_0(uX_0) I_1(uX_0)} \quad . \tag{4.2}$$

In this equation $G'(X_0)$ denotes the derivative of G with respect to X evaluated at $X = X_0$. The left side must now be expanded, which requires expanding α_1 in powers of W to first order. From Equation (3.12a), which we rewrite using the notation (4.1), we find

$$\alpha_1 = p\left[1 - \frac{W}{(d_0-\delta)^2}\right]^{1/2}\left[1 - \frac{W}{(d_0-\delta)^2 - v^2}\right]^{-1/2} \quad . \tag{4.3}$$

This expression could become very large, because of the inverse square root in the second factor. However, if $\alpha_1 b$ becomes large, the Bessel functions involving $\alpha_1 b$ on the lift of (4.2) will approximately cancel, and the factor α_1 in the denominator will make the left side small, while the right side is of order unity. Hence, there will not be a root of the equation in the region of large values of the inverse square root in α_1. Accordingly, the square roots in (4.3) may be expanded by binomial theorem, and only first order terms kept, without either producing or losing roots in the equivalent algebraic equation as compared to the original transcendental equation. When this expansion is carried out, there results:

$$\alpha_1 \sim p\left[1 + \frac{1}{2}\;\frac{Wv^2}{(d_0-\delta)^2\;(d_0-\delta)^2 - v^2}\right] \tag{4.4}$$

$$\sim p\,(1 + \epsilon).$$

The Bessel functions may be expanded as follows:

$$\frac{I_1(\alpha_1 b)}{I_1(pb)} \sim 1 + pb\epsilon\left[\frac{I_0(pb)}{I_1(pb)} - \frac{1}{pb}\right] \tag{4.5a}$$

$$\frac{I_0(pb)}{I_1(\alpha_1 b)} \sim 1 - pb\epsilon\;\frac{I_1(pb)}{I_0(pb)} \quad , \tag{4.5b}$$

and in the coefficients of ϵ it is permissible to replace pb by uX_0, since only second order errors are thereby committed. Upon inserting these results into the left hand side of (4.2) and simplifying, it reduces to:

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$$\frac{\eta^2 I_1(\alpha_1 b)}{p\alpha_1 I_1(pb)} \; \frac{I_o(pb)}{I_o(\alpha_1 b)} = 1 - \frac{\delta G'(X_o)}{uX_o I_o(uX_o) I_1(uX_o)} \quad . \tag{4.2}$$

In this equation $G'(X_o)$ denotes the derivative of G with respect to X evaluated at $X = X_o$. The left side must now be expanded, which requires expanding α_1 in powers of W to first order. From Equation (3.12a), which we rewrite using the notation (4.1), we find

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This expression could become very large, because of the inverse square root in the second factor. However, if $\alpha_1 b$ becomes large, the Bessel functions involving $\alpha_1 b$ on the lift of (4.2) will approximately cancel, and the factor α_1 in the denominator will make the left side small, while the right side is of order unity. Hence, there will not be a root of the equation in the region of large values of the inverse square root in α_1. Accordingly, the square roots in (4.3) may be expanded by binomial theorem, and only first order terms kept, without either producing or losing roots in the equivalent algebraic equation as compared to the original transcendental equation. When this expansion is carried out, there results:

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and in the coefficients of ϵ it is permissible to replace pb by uX_o, since only second order errors are thereby committed. Upon inserting these results into the left hand side of (4.2) and simplifying, it reduces to:

$$1 - \frac{W}{(d_o-\delta)^2-v^2} + \frac{1}{2}\,\frac{Wv^2}{(d_o-\delta)^2\left[(d_o-\delta)^2-v^2\right]}\,\frac{uX_o\left[I_o^2(uX_o)-I_1^2(uX_o)\right]}{I_o(uX_o)I_1(uX_o)} \quad . \tag{4.6}$$

Equating this to the right hand side of (4.2), dropping the ones, and slightly transforming, yields

$$\delta(d_o-\delta)^2\left|(d_o-\delta)^2-v^2\right| = \frac{WuX_oI_o(uX_o)I_1(uX_o)}{G'(X_o)}\left[(d_o-\delta)^2-\frac{1}{2}v^2\,\frac{uX_o\left[I_o^2(uX_o)-I_1^2(uX_o)\right]}{I_o(uX_o)I_1(uX_o)}\right] . \tag{4.7}$$

This equation is a quintic, since the left side is of fifth degree in δ. It may be written more simply by defining coefficients Q and R, defined by:

$$Q = -\frac{WuX_oI_o(uX_o)I_1(uX_o)}{G'(X_o)} \tag{4.8a}$$

$$R = \frac{uX_o\left|I_o^2(uX_o)-I_1^2(uX_o)\right|}{2I_o(uX_o)I_1(uX_o)} \tag{4.8b}$$

In terms of Q and R, the quintic (4.7) may be written in the form

$$\delta(d_o-\delta)^2\left[(d_o-\delta)^2-v^2\right] + Q\left[(d_o-\delta)^2-Rv^2\right] = 0 \,. \tag{4.9}$$

The coefficients Q and R are positive.

In the limit as the DC magnetic field and thereby v tends to infinity, Equation (4.9) should reduce to the cubic equation derived by Friedman[4]. The product QR is equal to Friedman's ζ, but the result of taking only the coefficient of v^2 in Equation (4.9) gives an equation in which the term corresponding to Friedman's ξ is missing. However, in the course of Friedman's work some terms of order higher than the first are kept, so this lack of agreement which corresponds to such a term should not be considered too serious.

A slightly simpler equation, whose roots have the same imaginary parts as have those of (4.9), is obtained by introducing a new variable y defined by

$$y = \delta - d_o . \tag{4.10}$$

The resulting quintic, fully written out, becomes:

$$y^5 + d_oy^4 - v^2y^3 + (Q - d_ov^2)y^2 - QRv^2 = 0. \tag{4.11}$$

The nature of the roots of this equation should now be considered.

Since (4.11) is a quintic with real coefficients, it has at least one real root. Denote the left side of (4.11) by $f(y)$. Then $f(-\infty) = -\infty$, while $f(\infty) = \infty$. Since both Q and R are positive, we have $f(0) = -QRv^2$ is negative, and hence there is at least one real positive root.

We shall next prove that $f(y)$ has at least two real negative roots, so that it can have at most two complex roots. The proof is based on the nature of the quantity R, as defined by Equation (4.8b). The important fact is that as uX_0, the argument of R, varies from zero to infinity, R decreases steadily from unity to one-half. This may be proven most simply by a direct numerical investigation. Now calculate $f(-v)$. A direct substitution shows that $f(-v) = Qv^2(1-R)$, and since R is between 1 and 1/2 while Q is positive, $f(-v)$ is positive. Accordingly, there is a root between $-\infty$ and $-v$, and another root between $-v$ and zero. These two roots may be expected to become infinite as v becomes infinite.

We have now proven that there are at most two complex roots of (4.11), which will be complex conjugates. Only one of these will correspond to an amplified wave. Accordingly, we have obtained the important result that the presence of a finite magnetic field does not produce new amplified waves. Thus the major effect, if any, must be in changing the propagation constant of the single amplified wave. An analysis of this effect will appear in a later paper.

The quintic (4.11) may be brought into the same form as Pierce's quintic. This requires only some transposition of terms and some division. However, more effects have been taken into account in (4.11) than in Pierce's corresponding equation. The cylindrical nature of the system and the solid nature of the beam have been considered. The longitudinal and transverse fields have been considered together since they are related by Maxwell's equations, while Pierce considers the two separately and introduces an arbitrary constant to denote their ratio. Finally, the nature of the approximations used in the theory have all been made explicit.

In summary and conclusion: An analysis has been performed of the effects of the finite DC magnetic field in traveling-wave tubes. The Maxwell equations have been solved in Section II, yielding the internal solution (2.18), and the external solution (2.19). The boundary conditions have been applied in Section III, winding up with the transcendental equation (3.15). In Section IV, this transcendental equation has been reduced to the quintic (4.11). It was shown that this quintic has at least three real roots, and hence that no new amplified waves are generated by the finiteness of the DC field and the resultant transverse motions of the electrons. The numerical analysis of (4.11) to give the dependence of the gain upon the magnetic field will be performed in a later paper.

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